2

Simplex Method and Transportation Problem

Structure

2.1. Introduction. Main limitation of graphical method in solving linear programming problem is that using this method, we can solve problems involving two variables only. In real life, we may have to solve the problems involving more than two variables. In such situation, we can't use this method. Simplex method is a technique with the help of which we can find solutions to such problems.

2.1.1. Objective. The objective of these contents is to provide some important results to the reader like:

(i) Simplex Method.

(ii) Duality.

(iii) Transportation Problem.

2.1.2. Keywords. Constraints, Unique Solution, Transportation.

2.2. Conditions for application of Simplex Method.

To apply this techniques the following two conditions must be satisfied

1. R.H.S of every constraint is equality must be non-negative. If it is negative in any in equality, it is made positive by multiplying both sides of inequality by (-1). For example if we are given the constraint $2x_1 - 5x_2 \geq -10$. Then we can rewrite it as $-2x_1 + 5x_2 \leq 10$) Note that when we multiply both sides by (-1), the sign of inequality changes.
2. Decision variables like x_1, x_2 would also be non-negative. If it is given that any decision variable is unrestricted in sign, it is expressed as difference of two non-negative variables, For example if it is give that x_3 is unrestricted in sight, we can write it as $x_3 = x_4 - x_5$.

Steps involued in simplex method

First write the objective function. It is either maximisation or minimisation.

For example, $Max\ Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ or $Min\ Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

Write the constraint inequalities with proper signs.

For example, $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1$ or $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \geq b_1$

and so on.

These in equalities are converted into equation by introducing slack and artificial variable. Slack variables now the left over quantity of the resources. Artificial variables have no real value. They are introduced first to solve the problem.

Important Note. If the in equations are of less than or equal to (≤) sign only slack variables are introduced. But if the in equations are of greater than or equal (≥ sign), both slack and artificial variables are introduced.

2.2.1. Example. Two given constraints are

$$2x_1 + 3x_2 \leq 60 \text{ and } 4x_1 + x_2 \geq 40$$

We will rewrite them as

$$2x_1 + 3x_2 + S_1 = 60 \text{ and } 4x_1 + x_2 - S_2 + A_1 = 40$$

In less then constraints, slack variables (S) will have positive sign and in more than constraints, they will have negative sign. These equations are then presented in the form of a matrix where format is shown below with the help of an example.

2.2.2. Example Maximise Z = 22 x_1 + 18 x_2

Subject to $x_1 + x_2 \leq 20$

$360\ x_1 + 240\ x_2 \leq 5760$

change the in equations into equations $x_1 + x_2 + S_1 = 20$

$360\ x_1 + 240x_2 + S_2 = 5760$

Table 1.

Basic	C_i	x_1	x_2	S_1	S_2	B	
S_1	0	1	1	1	0	20	Constraint values
S_2	0	360	240	0	1	5760	
C_j		22	18	0	0	Co-efficient values from constraint equation	
Zj		0	0	0	0		
Cj - Zj		22	18	0	0		

Cj is the contribution/unit of each variable shown in the objective function. Slack variables have zero contribution. Zj shows the total contribution of various variables at any given stage.

The row showing (Cj - Zj) is called the index row. This row shows how much profit is foregone by not producing one unit of a product etc.

Remember. An optimal solution is searched when all the values of index row become zero or negative. Now for finding the optimal solution, we consider two cases -(i) Maximisation case (ii) Minimisation case.

2.2.3. Maximisation Case. Let us reconsider the above example.

1) (Finding the initial feasible solution) our problem is

Max. $Z = 22x_1 + 18x_2$

Subject to $x_1 + x_2 \leq 20$

$360x_1 + 240x_2 \leq 5760$

or $x_1 + x_2 + S_1 = 20$ …(1)

$360x_1 + 240x_2 + S_2 = 5760$ …(2)

Initially, put $x_1 = 0$, $x_2 = 0$. So from (1) $S_1 = 20$ and from (2) $S_2 = 5760$.

In the initial solution, we assume that we are not producing any quantity of either of the products. So the resources remain fully unutilized. That whey in equation (1) we get $S_1 = 20$ and in (2) we get $S_2 = 5760$.

This solution is shown in the above table (in the term of matrix.)

Find the highest positive value in the row (Zj - Cj). The variable of the column to which this value corresponds will enter the solution. Divide the constraint values (b’s) by the element of this column to find the ratio (bij/aij). Choose the ratio which has minimum positive value and find the row of this ratio. The basic variable of this row will leave the solution and these above variable will replace this variable consider our example

Basic	C_i	x_1	x_2	S_1	S_2	b	b/a
S_1	0	1	1	1	0	20	20/1 = 20
S_2	0	360	240	0	1	5760	5760/360 = 16 →Outgoing variable
C_j		22	18	0	0		Co-efficient values from constraint equation
Zj		0	0	0	0		
Cj - Zj		22	18	0	0		

Incoming variable

So now x_1 will replace S_2

The element which belongs to both key column and key row is called key element. Now divide all elements of key row by key column like

Key row = 360 240 0 1 5760

Divide all the values by 360, we get

0 2/3 0 1/360 16

After using matrix operations, all other elements of key row are made equal to zero like

1st row	1	1	1	0	20
2nd row	1	2/3	0	1/360	16

R_1 ® R_1 - R_2 to get

1st row	0	1/3	1	-1/360	4
2nd row	1	2/3	0	1/360	16

After all these changes, the new matrices will be as follow

Table 2

		x_1	x_2	S_1	S_2	B	Ratio
0	S_1	0	1/3	1	-1/360	4	12 Key row
22	x_1	1	2/3	0	1/360	16	24
C_i		22	18	0	0	0*4+22*16=352 Total profit at this stage	
Zj		0	44/3	0	22/360		
Cj – Zj		0	10/3	0	-22/360		

Because still are positive value remains in the (C_j - Z_j) row, so we have get to obtain optimal solution. Now we will repeat steps 2 and 3 and repeat them till all the values in the (C_j - Z_j) row become be zero or negative.

The new table will be as follows:

Table 3

		x_1	x_2	S_1	S_2	b
18	x_2	0	1	3	-1/120	12
22	x_1	1	0	-2	1/120	8
C_i		22	18	0	0	18*12+22*8=392
Zj		22	18	10	4/120	Total profit at this stage
Cj – Zj		0	0	-10	-4/120	

Now there is no positive value in the index row, so we have obtained optimal solution. The optimal solution is x_1 = 8, x_2 = 12 and maximum profit Z = Rs. 392 (obtained from the resources column)

2.2.4. Example. A firm produces three products A, B and C, each of which passes through three departments : Fabrication, Finishing and Packaging. Each unit of product A requires 3, 4 and 2; a unit of product B requires 5, 4 and 4, while each unit of product C requires 2, 4 and 5 hours respectively in the three departments. Every day, 60 hours are available in the fabrication department, 72 hours in the finishing department and 100 hours in the packaging department. The unit contribution of product A is Rs 5, of product B is Rs. 10, and of product C is Rs. 8.

Required :

a) Formulate the problem as an LPP and determine the number of units of each of the products, that should be made each day to maximise the total contribution. Also determine if any capacity would remain unutilized.

Solution. Let x_1, x_2 and x_3 represent the number of units of products A, B and C respectively. The given problem can be expressed as a LPP as follows :

Maximise $Z = 5x_1 + 10x_2 + 8x_3$ Contribution

Subject to $3x_1 + 5x_2 + 2x_3 \leq 60$ Fabrication hours

$4x_1 + 4x_2 + 4x_3 \leq 72$ Finishing hours

$2x_1 + 4x_2 + 5x_3 \leq 100$ Packaging hours

$x_1, x_2, x_3 \geq 0$

Introducing slack variables, the augmented problem can be written as

Maximise $Z = 5x_1 + 10x_2 + 8x_3 + 0S_1 + 0S_2 + 0S_3$

Subject to

$$3x_1 + 5x_2 + 2x_3 + S_1 = 60$$

$$4x_1 + 4x_2 + 4x_3 + S_2 = 72$$

$$2x_1 + 4x_2 + 5x_3 + S_3 = 100$$

$$x_1, x_2, x_3, S_1, S_2, S_3 \geq 0$$

The solution to the problem using simplex algorithm is contained in Tables 1 to 3.

Simplex Table 1: Initial Solution

Basic		x_1	x_2	x_3	S_1	S_2	S_3	b	Ratio b_i/a_{ij}	
S_1	0	3	2	5*	1	0	0	60	12	Outgoing variable (key row)
S_2	0	4	4	4	0	1	0	72	18	
S_3	0	2	4	5	0	0	1	100	15	
C_i		5	10	8	0	0	0			
Zj		0	0	0	0	0	0			
Cj - Zj		5	10	8	0	0	0			

5* is the key element

Simplex Table 2: Non-optimal Solution

Basic		x_1	x_2	x_3	S_1	S_2	S_3	b	Ratio b_i/a_{ij}	
x_2	10	3/5	1	2/5	1/5	0	0	12	30	Outgoing variable (key row)
S_2	0	8/5	0	12/5	-4/5	1	0	24	10	
S_3	0	-2/5	0	17/5	-4/5	0	1	52	260/17	
C_i		5	10	8	0	0	0			
Zj		6	10	4	2	0	0			
Cj - Zj		-1	0	4	-2	0	0			

Simplex Table 3: Optimal Solution

Basic		x_1	x_2	x_3	S_1	S_2	S_3	b_i
x_2	10	1/3	1	0	1/3	-1/6	0	8
x_3	8	2/3	0	1	-1/3	5/12	0	10
S_3	0	-8/3	0	0	1/3	-17/12	1	18
C_i		5	10	8	0	0	0	
Solution		0	8	10	2	0	18	
Cj - Zj		-11/3	0	0	-2/3	-5/3	0	

According to the Simplex Table 3, the optimal solution is : $x_1 = 0$, $x_2 = 8$, $x_3 = 10$. Thus, it calls for producing 8 and 10 units of products B and C respectively, each day. This mix would yield a contribution of 5 * 0 + 10* 8 + 8 *10 = Rs. 160. S_3 being equal to 18, an equal number of hours shall remain unutilized in the packaging department.

2.2.5. Example. Solve the following L.P.P.

Maximise $Z = 40000\,x_1 + 55000x_2$

Subject to $1000\,x_1 + 1500x_2 \leq 20000$

$x_1 \leq 12$

$x_2 \geq 5$

$x_1, x_2 \geq 0$

Solution. By changing the inequations into equations by adding surplus and artificial variables, the form of the problem is changed as :

Maximise $Z = 40000x_1 + 15000x_2 + 0.S_1 + 0S_2 + 0S_3 - M.A.$

Subject to $1000x_1 + 1500x_2 + S_1 = 2000$

$x_1 + S_2 = 12$

$x_2 - S_3 + A_1 = 5$

$x_1, x_2, S_1, S_2, S_3, A_1 \geq 0$

The solution to this problem is shown in tables 1 to 3

Simplex Table 1: Initial Solution

Basic		x_1	x_2	S_1	S_2	S_3	A_1	b_i	b_i/a_{ij}
S_1	0	1000	1500	1	0	0	0	20000	40/3
S_2	0	1	0	0	1	0	0	12	-
A_1	-M	0	1*	0	0	-1	1	5	5 (key row)
C_j		40000	55000	0	0	0	-M		
Z_j		0	-M	0	0	M	-M	-5M	
$C_j - Z_j$		40000	55000+M	0	0	-M	0		

(Incoming variable)
Key column

Simplex Table 2: Non-optimal Solution

Basic		x_1	x_2	S_1	S_2	S_3	A_1	b	b_i/a_{ij}
S_1	0	1000	0	1	0	1500*	-1500	12500	1250/1500 (key row)
S_2	0	1	0	0	1	0	0	12	-
x_2	55000	0	1	0	0	-1	1	5	-
C_j		40000	55000	0	0	0	-M		
Z_j		0	55000	0	0	-55000	55000	275000	
$C_j - Z_j$		40000	0	0	0	55000	-M-55000		

(key column)

*key element

Simplex Table 3: Non-optimal Solution

Basic		x_1	x_2	S_1	S_2	S_3	A_1	B	b_i/a_{ij}
S_3	0	2/3	0	1/1500	0	1	-1	25/3	25/212 (key row)
S_2	0	1	0	0	1	0	0	12	-
x_2	55000	2/3	1	1/1500	0	0	0	40/3	20
C_j		40000	55000	0	0	0	-M		
Z_j		11000/3	55000	110/3	0	0	0	2200000/3	
$C_j - Z_j$		10000/3	0	-110/3	0	0	-M		
		(key column)							

Simplex Table 4: Optimal Solution

Basic		x_1	x_2	S_1	S_2	S_3	A_1	B
S_3	0	0	0	1/1500	-2/3	1	-1	1/3
x_1	40000	1	0	0	1	0	0	12
x_2	55000	0	1	1/1500	-2/3	0	0	16/3
C_j		40000	55000	0	0	0	-M	
Z_j		40000	55000	110/3	10000/3	0	0	2320000/3
$C_j - Z_j$		0	0	-110/3	-10000/3	0	-M	

So optimal solution is $x_1 = 12$, $x_2 = 16/3$ and Z = 2320000/3.

2.2.6. Minimization Case.

Steps involved in finding the minimum value of objective functions are same as in case of maximization. Same fundamental differences should be taken case of which are as follows :

1. In the table showing initial solution, we will take highest negative value not the highest positive value. The column which has this value is the key column.
2. In problems of minimisation, if we use artificial variables then they will have a weight of +M whereas in problems of maximization, they have negative weight -M.
3. While going for optimal solution, these artificial variables leave the solution. If they are in the solution in the final table, it means that the given problem has no feasible solution.
4. When all the values in the index row are zero a positive, optimal solution is reached.

2.2.7. Example. To improve the productivity of land, a framer is advised to use at least 4800 kg. of phosphate fertilizer and not less than 7200 kg. of nitrogen fertilizer. There are two sources to object these fertilizers mixture A and B. Both of these are available in bags of 100 kg. each and their cost per bag are Rs. 40 and Rs. 24 respectively. Mixture A contains 20 kg. phosphate and 80 kg. nitrogen while their respective quantities in mixture B are 80 kg. and 50 kg. Formulate this as an LPP and determine how many bags of each type of mixture the farmer should buy in order to obtain the required fertilizer at minimum cost.

Solution. Let x1 be number of bags of mixture A and x2 be the number of bags of mixture B. So now the problem can be written as

Minimise $Z = 40x_1 + 24x_2$ Total cost

Subject to

$20x_1 + 50x_2 \geq 4800$ Phosphate Requirement

$80x_1 + 50x_2 \geq 7200$ Nitrogen Requirement

$x_1, x_2 \geq 0$

After introducing the slack + artificial variables, the above problem can be rewritten as :

Minimise $Z = 40x_1 + 24x_2 + 0S_1 + 0S_2 + MA_1 + MA_2$

Subject to

$20x_1 + 50x_2 - S_1 + A_1 = 4800$

$80x_1 + 50x_2 - S_2 + A_2 = 7200$

Simplex Table 1: Initial Solution

Basic		x_1	x_2	S_1	S_2	A_1	A_2	b_i	b_i/a_{ij}
A_1	M	20	50*	-1	0	1	0	4800	96 (key row)
A_2	M	80	50	0	-1	0	1	7200	144
C_j		40	24	0	0	M	M		
Z_j		100M	100M	-M	-M	M	M	12000M	
$C_j - Z_j$		40-100M	24-100M	M	M	0	0		

(Key column)

*key element

Simplex Table 2: Non-optimal Solution

Basic		x_1	x_2	S_1	S_2	A_1	A_2	b_i	b_i/a_{ij}
x_2	24	2/5	1	-1/50	0	1/50	0	96	240
A_2	M	60	0	1	-1	-1	1	2400	40 (key row)
C_j		40	24	0	0	M	M		
Z_j		48/5+60M	24	M-24/50	-M	-M+24/50	M	2304+	
$C_j - Z_j$		152/2-60M	0	12/25-M	M	2M-12/25	0	2400M	

(Key column)

Simplex Table 3: Non-optimal Solution

Basic		x_1	x_2	S_1	S_2	A_1	A_2	b_i	b_i/a_{ij}
x_2	24	0	1	-2/75	1/150	2/75	-1/150	80	-3000
x_1	40	1	0	1/60*	-1/60	-1/60	1/60	40	2400(key row)
C_j		40	24	0	0	M	M		
Z_j		40	24	2/75	-38/75	-2/75	-38/75	3520	
$C_j - Z_j$		0	0	-2/75	38/75	M+2/75	M+38/75		
				Key column					

Simplex Table 4: Optimal Solution

Basic		x_1	x_2	S_1	S_2	A_1	A_2	b_i
x_2	24	8/5	1	0	-1/150	0	1/50	144
S_1	0	60	0	1	-1	-1	1	2400
C_j		40	24	0	0	M	M	
Z_j		192/5	24	0	-12/25	M	12/25	3456
$C_j - Z_j$		8/5	0	0	12/25	0	M-12/25	

Since all the values of the index row are zero or positive, so we have got optimal solution. The optimal solution $x_2 = 144$, $x_1 = 0$ and Z = Rs. 3456.

2.2.8. Example. A finished product must weigh exactly 150 grams. The two raw materials used in manufacturing the product are A, with a cost of Rs. 2 per unit and B with a cost of Rs. 8 per unit. At least 14 units of B not more than 20 units of A must be used. Each unit of A and B weighs 5 and 10 grams respectively. How much of each type of raw material should be used for each unit of the final product in order to minimise the cost ? Use Simplex method.

Solution. The given problem can be expressed as LPP as

Minimise $Z = 2x_1 + 8x_3$

Subject to $5x_1 + 10x_3 = 150$

$x_1 \leq 20$

$x_2 \geq 14$

$x_1, x_2 \geq 0$

Substituting $x_2 = 14 + x_3$ and introducing necessary slack and artificial variables, we have, Minimise $Z = 2x_1 + 8x_3 + 112 + MA_1 + 0x_4$

Subject to $5x_1 + 10x_3 + A_1 = 10$

$x_1 + x_4 = 20$

$x_1, x_3, x_4, A_1 \geq 0$

The solution is contained in the following tables.

Simplex Table 1: Initial Solution

Basic		x_1	x_3	A_1	x_4	b_i	b_i/a_{ij}
A_1	M	5	10*	1	0	10	1(key row)
x_4	0	1	0	0	1	20	-
C_j		2	8	M	0		
Z_j		5M	10M	M	0	10M	
$C_j - Z_j$		2-5M	8-10M	0	0		

(Key column)

*key element

Simplex Table 2: Non-optimal Solution

Basic		x_1	x_3	A_1	x_4	b_i	b_i/a_{ij}
x_3	8	½*	1	1/10	0	1	2(key row)
x_4	0	1	0	0	1	20	20
C_j		2	8	M	0		
Z_j		4	8	8/10	0	8	
$C_j - Z_j$		-2	0	M-8/10	0		

(Key column)

*key element

Simplex Table 3: Optimal Solution

Basic		x_1	x_3	A_1	x_4	b_i
x_1	2	1	2	1/5	0	2
x_4	0	0	-2	-1/5	1	18
C_j		2	8	M	0	
Z_j		0	4	2/5	0	4
$C_j - Z_j$		0	4	M-2/5	0	

Thus, the optimal solution is : x_1 = 2 units, x_3 = 14 + 0 = 14 units, total cost = 2 *2 + 8* 14 = Rs. 116.

2.2.9. Example. A company produces three products, P_1, P_2 and P_3 from two raw materials A and B, and labour L. One, unit of product P_1 requires one unit of A, 3 units of B and 2 units of L. One unit of product P_2 requires 2 units of A and B each, and 3 units of L, while one units of P_3 needs 2 units of A, 6 units of B and 4 units of L. The company has a daily availability of 8 units of A, 12 units of B and 12 units of L. It is further known that the unit contribution margin for the products is Rs. 3, 2 and 5 respectively for P_1, P_2 and P_3. Formulate this problem as a linear programming problem, and then solve it to determine the optimum product mix. Is the solution obtained by you unique ? Identify an alternate optimum solution, if any.

Solution. If x_1, x_2 and x_3 be the output of the products P_1, P_2 and P_3, respectively, we may express the linear programming formulation as follows :

Maximise $Z = 3x_1 + 2x_2 + 5x_3$ Contribution

Subject to $x_1 + 2x_2 + 2x_3 \leq 8$ Material A

$3x_1 + 2x_2 + 6x_3 \leq 12$ Material B

$2x_1 + 3x_2 + 4x_3 \leq 12$ Labour

$x_1, x_2, x_3 \geq 0$

Introducing slack variables S_1, S_2 and S_3, we may write the problem as follows :

Maximise $Z = 3x_1 + 2x_2 + 5x_3 + 0S_1 + 0S_2 + 0S_3$

Subject to $x_1 + 2x_2 + 2x_3 + S_1 = 8$

$3x_1 + 2x_2 + 6x_3 + S_2 = 12$

$2x_1 + 3x_2 + 4x_3 + S_3 = 12$

$x_1, x_2, x_3, S_1, S_2, S_3 \geq 0$

Simplex Table 1: Initial Solution

Basic		x_1	x_2	x_3	S_1	S_1	S_3	b_i	b_i/a_{ij}
S_1	0	1	2	2	1	0	0	8	4
S_2	0	3	2	6*	0	1	0	12	2(key row)
S_3	0	2	3	4	0	0	1	12	3
C_j		3	2	5	0	0	0		
Z_j		0	0	0	0	0	0		
$C_j - Z_j$		3	2	5	0	0	0		
				Key column					

Simplex Table 2: Non-optimal Solution

Basic		x_1	x_2	x_3	S_1	S_1	S_3	b_i	b_i/a_{ij}
S_1	0	0	4/3	0	1	-1/3	0	4	-
x_3	5	½*	1/3	1	0	1/6	0	2	4(key row)
S_3	0	0	5/3	0	0	-2/3	1	4	-
C_j		3	2	5	0	0	0		
Z_j		5/2	5/3	5	4	5/6	4	10	
$C_j - Z_j$		½	1/3	0	0	-5/6	0		
		Key column							

Simplex Table 3: Optimal Solution

Basic		x_1	x_2	x_3	S_1	S_1	S_3	b_i	b_i/a_{ij}
S_1	0	0	4/3	0	1	-1/3	0	4	3
X_1	3	1	2/3	2	0	1/6	0	4	6
S_3	0	0	5/3*	0	0	-2/3	1	4	12/5(key row)
C_j		3	2	5	0	0	0		
Z_j		3	2	6	4	1	0	12	
$C_j - Z_j$		0	0	-1	0	-1	0		

The solution contained in Table 3 is optimal with $x_1 = 4$, $x_2 = x_3 = 0$ and Z = 12. However, it is not unique since x_2, a non-basic variable, has $C_j - Z_j$ equal to zero. The problem, thus, has an alternate optimal solution. To obtain this, we revise the solution in Table 3with x_2 as the entering variable. It is given in Simplex Table 4.

Simplex Table 3: Alternate Optimal Solution

Basic		x_1	x_2	x_3	S_1	S_1	S_3	b_i
S_1	0	0	0	0	1	1/5	-4/5	4/5
x_1	3	1	0	2	0	3/5	-2/5	12/5
x_2	2	0	1	0	0	-2/5	3/5	12/5
C_j		3	2	5	0	0	0	
Z_j		12/5	12/5	0	4/5	0	0	12
$C_j - Z_j$		0	0	-1	0	-1	0	

2.3. Duality in Linear Programming.

For every linear programming problem there is another linear programming problem which is related to it and which is obtained from it. First problem is called primal and second is called its dual.

2.3.1. Rules for obtaining dual from primal :-

1) Co-efficients of variables in objective function of primal become constraint values in the dual and constraint values in the primal becomes coefficients of variables in the objective function.

2) If he primal is of maximization type, dual is of minimisation type and if primal is of minimisation dual is of maximisation.
3) Co-efficient of first column of constraints of primal because co-efficient of first row of dual, second column becomes second row and so on.
4) Direction of constraint in equations is also changed. It is primal they are of $\leq$ type, in dual they will be $\geq$ type.

Besides these, the following things should also be kept in mind :

i. All the variables in the dual must be non-negative.

ii. If the dual is of minimisation objective, all the constraints must be of £ type and if it is of minimisation, all the constraints must be of $\geq$ type. In any dual, we can't have mixed constraints.

Mathematically, change from primal to dual can be shown, with the help of an example.

2.3.2. Example. For the LPP given below, write the dual.

Maximise $Z = 40x_1 + 35x_2$

Subject to $2x_1 + 3x_2 \leq 60$

$4x_1 + 3x_2 \leq 96$

$x_1, x_2 \geq 0$

Solution. In accordance with above, its dual shall be

Minimise $G = 60y_1 + 96y_2$,

Subject to $2y_1 + 4y_2 \geq 40$

$3y_1 + 3y_2 \geq 35$

$y_1, y_2 \geq 0$

2.3.3. Obtaining Dual of LPP with Mixed Restrictions

Sometimes a given LPP has mixed restrictions so that the inequalities given are not all in the right direction. In such a case, we should convert the inequalities in the wrong direction into those in the right direction. Similarly, if an equation is given in respect of a certain constraint, it should also be converted into inequality. To understand fully, consider the following examples.

2.3.4. Example. Write the dual of the following LPP.

Minimise $Z = 10x_1 + 20x_2$

Subject to $3x_1 + 2x_2 \geq 18$

$x_1 + 3x_2 \geq 8$

$2x_1 - x_2 \leq 6$

$x_1, x_2 \geq 0$

Solution. Here, the first two inequalities are in the right direction (being ≥ type with a minimisation type of objective function) while the third one is not. Multiplying both sides by -1, this can be written as $-2x_1 + x_2 \geq 6$. Now, we can write the primal and dual as follows :

	Primal		Dual
Minimise	$Z = 10x_1 + 20x_2$	Maximise	$G = 18y_1 + 8y_2 - 6y_3$
Subject to		Subject to	
	$3x_1 + 2x_2 \geq 18$		$3y_1 + y_2 - 2y_3 \leq 10$
	$x_1 + 3x_2 \geq 8$		$2y_1 + 3y_2 + y_3 \leq 20$
	$-2x_1 + x_2 \geq -6$		
	$x_1, x_2 \geq 0$		$y_1, y_2, y_3 \geq 0$

2.3.5. Example. Obtain the dual of the LPP given here :

Maximise $Z = 8x_1 + 10x_2 + 5x_3$

Subject to

$x_1 - x_3 \leq 4$

$2x_1 + 4x_2 \leq 12$

$x_1 + x_2 + x_3 \geq 2$

$3x_1 + 2x_2 - x_3 = 8$

$x_1, x_2, x_3 \geq 0$

Solution. We shall first consider the constraints.

Constraints 1 and 2 ; Since they are both of the type ≤ , we do not need to modify them.

Constraint 3 : This is of type ≥. Therefore, we can convert it into ≤ type by multiplying both sides by -1 to become $-x_1 - x_2 - x_3 \leq -2$.

Constraint 4 : It is in the form of an equation. An equation, mathematically, can be

represented by a part of inequalities: one of ≤ type and the other of ≥ type. The given constraint can be expressed as

$$3x_1 + 2x_2 - x_3 \leq 8$$

$$3x_1 + 2x_2 - x_3 \geq 8$$

The second of these can again be converted into type ≤ by multiplying by -1 on both sides. Thus it can be written as $-3x_1 - 2x_2 + x_3 \leq -8$.

Now we can write the primal and the dual as follows :

	Primal		Dual
Maximise	$Z = 8x_1 + 10x_2 + 5x_3$	Minimise	$G = 4y_1 + 12y_2 - 2y_3 + 8y_4 - 8y_5$

Subject to

$x_1 - x_3 \leq 4$

$2x_1 + 4x_2 \leq 12$

$-x_1 - x_2 - x_3 \leq -2$

$3x_1 + 2x_2 + x_3 \leq -8$

$-3x_1 - 2x_2 + x_3 \leq -8$

$x_1, x_2, x_3 \geq 0$

Subject to

$y_1 + 2y_2 - y_3 + 3y_4 - 3y_5 \geq 8$

$4y_2 - y_3 + 2y_4 - 2y_5 \geq 10$

$-y_1 - y_3 - y_4 + y_5 \geq 5$

$y_1, y_2, y_3, y_4, y_5 \geq 0$

One point needs mention here. We know that corresponding to an n-variable, m-constraint primal problem, there would be m-variable, n-constraint dual problem. For this example involving three variables and four constraints, the dual should have four variables and three constraints. But we observe that the dual that we have obtained contains five variables. The seeming inconsistency can be resolved by expressing $(y_4 - y_5) = y_6$, a variable unrestricted in sign. Thus, although, y_4 and y_5 are both non-negative, their difference could be greater than, less than, or equal to zero. The dual can be rewritten as follows :

Minimse $\quad G = 4y_1 + 12y_2 - 2y_3 + 8y_6$

Subject to

$y_1 + 2y_2 - y_3 + 3y_6 \geq 8$

$4y_2 - y_3 + 2y_6 \geq 10$

$-y_1 - y_3 - y_6 \geq 5$

$y_1, y_2, y_3 \geq 0$, y_6 unrestricted in sign

Thus, whenever a constraint in the primal involves an equality sign, its corresponding dual variable shall be unrestricted in sign. Similarly, an unrestricted variable in the primal would imply that the corresponding constraint shall bear the = sign.

2.3.6. Example. Obtain the dual of the following LPP :

Maximise $\quad Z = 3x_1 + 5x_2 + 7x_3$

Subject to $\quad x_1 + x_2 + 3x_3 \leq 10$

$4x_1 - x_2 + 2x_3 \geq 15$

$x_1, x_2 \geq 0$, x_3 unrestricted in sign

Solution. First of all, we should convert the second restriction into the type $\leq$. This results in $-4x_1 + x_2 - 2x_3 \leq -15$.

Next, we replace the variable x_3 by the difference of two non-negative variables, say, x_4 and x_5. This yields the primal problem corresponding to which dual can be written, as shown against it.

Primal	Dual
Maximise $Z = 3x_1 + 5x_2 + 7x_4 - 7x_5$	Minimise $G = 10y_1 - 15y_2$
Subject to	Subject to
$x_1 + x_2 + 3x_4 - 3x_5 \leq 10$	$y_1 - 4y_2 \geq 3$
$-4x_1 + x_2 - 2x_4 + 2x_5 \leq -15$	$y_1 + y_2 \geq 5$
$x_1, x_2, x_4, x_5 \geq 0$	$3y_1 - 2y_2 \geq 7$
	$-3y_1 + 2y_2 \geq -7$
	$y_1, y_2 \geq 0$

The fourth constraint of the dual can be expressed as $3y_1 - 2y_2 \leq 7$. Now, combining the third and the fourth constraints, we get $3y_1 - 2y_2 = 7$. The dual can be expressed as follows :

Minimise $G = 10y_1 - 15y_2$

Subject to

$$y_1 - 4y_2 \geq 3$$

$$y_1 + y_2 \geq 5$$

$$3y_1 - 2y_2 = 7$$

$$y_1, y_2 \geq 0$$

The symmetrical relationship between the primal and dual problems, assuming the primal to be a 'maximisation' problem is depicted in the Chart.

Primal	Dual
Maximization	Minimisation
No. of variables	No. of constraints
No. of constraints	No. of variables
$\leq$ type constraint	Non-negative variable
= type constraint	Unrestricted variable
Unrestricted variable	= type constraint
Objective function coefficient for j^{th} variable	RHS constant for the j^{th} constraint
RHS constant for j^{th} constraint	Objective function coefficient for j^{th} variable
Coefficient (a_{ij}) for j^{th} variable in i^{th} constraint	Coefficient (a_{ij}) for i^{th} variable in j^{th} constraint

Comparing the Optimal Solutions of the Primal and Dual

Since the dual of a given primal problem is derived from and related to it, it is natural to except that the (optimal) solutions to the two problems shall be related to each other in the same way. To understand this, let us consider the following primal and dual problems again and compare their optimal solutions.

Primal	Dual
Maximise $Z = 40x_1 + 35x_2$	Minimise $G = 60y_1 + 96y_2$
Subject to	Subject to
$2x_1 + 3x_2 \leq 60$	$2y_1 + 4y_2 \geq 40$
$4x_1 + 3x_2 \leq 96$	$3y_1 + 3y_2 \geq 35$
$x_1, x_2 \geq 0$	$y_1, y_2 \geq 0$

The simplex table containing optimal solution to the primal problem is reproduced.

Simplex Table 1: Initial Solution

Basic		y_1	y_2	S_1	S_2	A_1	A_2	b_i	b_i/a_{ij}
A_1	M	2	4*	-1	0	1	0	40	10(key row)
A_2	M	3	3	0	-1	0	1	35	35/3
C_j		60	96	0	0	M	M		
Z_j		5M	7M	-M	-M	M	M		
$C_j - Z_j$		60-5M	96-7M	M	M	0	0		

Simplex Table 2: Non-optimal Solution

Basic		y_1	y_2	S_1	S_2	A_1	A_2	b_i	b_i/a_{ij}
y_2	96	1/2	1	-1/4	0	1/4	0	10	20
A_2	M	3/2*	0	3/4	-1	-3/4	1	5	10/3 (key row)
C_j		60	96	0	0	M	M		
Z_j		48+3M/2	96	-24+3M/4	-M	24-3M/4	M		
$C_j - Z_j$		12-3M/2	0	24-3M/4	M	7M/4-24	0		

Simplex Table 2: Non-optimal Solution

Basic		y_1	y_2	S_1	S_2	A_1	A_2	b_i
y_2	96	0	1	-1/2	1/3	½	0	25/3
y_1	60	1	0	1/2	-2/3	-1/2	1	10/3
C_j		60	96	0	0	M	M	
Z_j		60	96	-18	-8	18	8	
$C_j - Z_j$		0	0	18	8	M-18	M-8	

Before comparing the solutions, it may be noted that there is a correspondence between variables of the primal and the dual problems. The structural variable x_1 in the primal, corresponds to the surplus variable S_1 in the dual, while the variable x_2 corresponds to S_2, the other surplus variable in the dual. In a similar way, the structural variables y_1 and y_2 in the dual correspond to the slack variables S_1 and S_2 respectively of the primal.

A comparison of the optimal solutions to the primal and the dual, and some observations follow.

a) The objective function values of both the problems are the same. This with $x_1 = 18$ and $x_2 = 8$, Z equals 40*18 + 35* 8 = 1000. Similarly, with $y_1 = 10/3$ and $y_2 = 25/3$, the value of G would be 60 *10/3 + 96 *25/3 = 1000.

b) The numerical value of each of the variables in the optimal solution to the primal is equal to the value of its corresponding variable in the dual contained in the C_j - Z_j row. Thus, in the primal problem, $x_1 = 18$ and $x_2 = 8$, whereas in the dual $S_1 = 18$ and $S_2 = 8$ (in the C_j - Z_j row).

Similarly, the numerical value of each of the variables in the optimal solution to the dual is equal to the value of its corresponding variable in the primal, as contained in the C_j - Z_j row of it. Thus, $y_1 = 10/3$ and $y_2 = 25/3$ in the dual, and $S_1 = 10/3$ and $S_2 = 25/3$ (note that we consider only the absolute values) in the primal. Of course, we do not consider artificial variables because they do not correspond to any variables in the primal, and are introduced for a specific, limited purpose only.

Clearly then, if feasible solutions exist for both the primal and the dual problems then both problems have optimal solutions of which objective function values are equal. A peripheral relationship between them is that if one problem has an unbounded solution, its dual has no feasible solution.

Further, the optimal solution to the dual can be read from the optimal solution of the primal, and vice versa. The primal and dual need not both be solved, therefore, to obtain the solution. This offers a big computational advantage in some situations. For instance, if the primal problem is a minimization one involving, say 3, variables and 7 constraints, its solution would pose a big problem because a large number of surplus and artificial variables would have to be introduced. The number of iterations required for obtaining the answer would also be large. On the counter, the dual, with 7 variables and 3 constraints can be solved comparatively much more easily.

2.4. Transportation Problems.

If a company manufactures one products in two or more factories and has two or more main go-downs from where the product can be supplied to the customers, then the company has to decide how much quantity of each factory should be transported to each of the godown so that total transportation cost is minimised. Though we have other method to solve this problem, yet linear programming can also help in solving the transportation problems.

For example, a company has three plants P_1, P_2, P_3, and three warehouses W_1, W_2 and W_3 . Now various entities and costs can be shown in the form of the following matrix.

To / From	W_1	W_2	W_3	Supply
P_1	x_{11} C_{11}	x_{12} C_{12}	x_{13} C_{13}	S_1
P_2	x_{21} C_{21}	x_{22} C_{22}	x_{23} C_{23}	S_2
P_3	x_{31} C_{31}	x_{32} C_{32}	x_{33} C_{33}	S_3
Demand	D_1	D_2	D_3	

It is assumed that total supply = total demand.

In the above matrix c_{ij} represents transportation cost /unit from factory i to warehouse j and x_{ij} represents quantity (in units) transported from factory i to warehouse j.

Now

Objective function is

Minimise $Z = x_{11}\, c_{11} + x_{12}\, c_{12} + x_{13}\, c_{13} + x_{21}\, c_{21} + x_{22}\, c_{22} + x_{23}\, c_{23} + x_{31}\, c_{31}+ x_{32}\, c_{32} + x_{33}\, c_{33}$

Subject to $x_{11} + x_{12} + x_{13} = S_1$

$x_{21} + x_{22} + x_{23} = S_2$ Supply constraints

$x_{31} + x_{32} + x_{33} = S_3$

$x_{11} + x_{21} + x_{31} = D_1$

$x_{12} + x_{22} + x_{32} = D_2$ Demand constraints.

$x_{13} + x_{23} + x_{33} = D_3$

$x_{ij} \geq 0$ for i = 1, 2, 3 and j = 1, 2, 3 .

As we can see that if we use simplex method to solve the above problem, having 9 decision variables and 6 constraints, it will be a long process and so this method is not generally used to solve transportation problems. So we shall confine ourselves to graphical method for solving these problems. In other words, we will have only two decision variables (say x and y).

2.4.1. Example. A company manufacturing a product has two plants P_1 and P_2 having weekly capacities 100 and 60 units respectively. The cars are transported to three godowns w_1, w_2 and w_3 whose weekly requirements are 70, 50 and 40 units respectively. The transportation costs (Rs./unit) are as given below:

P_1-w_1 = 5 , P_1-w_2 = 4 , P_1-w_3= 3, P_2-w_1= 4, P_2-w_2=2, P_3-w_2 =5.

Solve the above transportation problem so as to minimise total transportation costs.

Solution. First consider plant P_1. Let x and y be the units transported from P_1 to w_1 and w_2. Now we complete the matrix in the following form

To From	w_1 cost/Qty. unit	w_2 cost/Qty. unit	w_3 cost/Qty. unit	Supply
P_1	5 x	4 y	3 (100-x-y)	100
P_2	4 (70-x)	2 (50-y)	5 (x+y-60)	60
Demand	70	50	40	160

Now total cost = 5x+4(70-x) + 4y + 2(50-y) +3(100-x-y) +5(x+y-60)

= 5x+280 -4x+4y+100-2y+300-3x-3y+5x+5y-300 = 3x+4y+380

So objective function is

Min. Z = 3x+4y+380

Subject to the constraints

(i) In first row $100-x-y \geq 0$ so $x+y \leq 100$

(ii) In 2nd row $70-x \geq 0$, $50-y \geq 0$ and $x+y-60 \geq 0$

So $x \leq 70$, $y \leq 50$ and $x+y \geq 60$

So we have 4 inequations

(i) $x+y \leq 100$ (ii) $x \leq 70$ (iii) $y \leq 50$ and (iv) $x+y \geq 60$.

Plotting these values on the graph, we get the following feasible region.

The feasible region lies in the area covered by the polygon ABCDE. We also know that optimal solution lies at one of the vertices. So now we find the values of x, y and z at these points.

Points	**x**	**y**	**Z = 3x+4y+380**
A	60	0	60*3+4*0+380 = 560
B	70	0	70*3+4*0+380 = 590
C	70	30	70*3+30*4+380 = 710
D	50	50	50*5+50*4+380 = 830
E	10	50	10*5+50*4+380 = 630

Since the minimum value of Z is Rs. 560 at A, so optimal values of x and y are x = 60, y = 0.

So the optimal transportation schedule is

From P_1, 60 units will be transported to w_1 and 40 units to w_3 .

From P_2, 10 units will be transported to w_1 and 50 units to w_2 .

2.5. Check Your Progress.

Solve the following linear programming using simplex method.

1. Maximise $Z = 7x_1 + 14x_2$

   Subject to the constraints

   $3x_1 + 2x_2 \leq 36$

   $x_1 + 4x_2 \leq 10$

   $x_1, x_2 \geq 0$

2. Maximise $Z = 20x_1 + 30x_2 + 5x_3$,

   Subject to

   $4x_1 + 3x_2 + x_3 \leq 40$

   $2x_1 + 5x_2 \leq 28$

   $8x_1 + 2x_2 \leq 36$

   and $x_1, x_2, x_3 \geq 0$

3. Maximise $Z = 10x_1 + 20x_2$,

   Subject to

   $2x_1 + 5x_2 \geq 50$

   $4x_1 + x_2 \leq 28$

   $x_1, x_2 \geq 0$

4. Minimise $Z = 6x_1 + 4x_2$

   Subject to

   $3x_1 + 0.5x_2 \geq 12$

   $2x_1 + x_2 \geq 16$

   $x_1, x_2 \geq 0$

5. Using two-phase Method, solve the following problem :

   Minimise $150x_1 + 150x_2 + 100x_3$,

   Subject to

   $2x_1 + 3x_2 + x_3 \geq 4$

   $3x_1 + 2x_2 + x_3 \geq 3$

   and $x_1, x_2, x_3 \geq 0$

6. Solve the following LPP :

Minimise $Z = 100x_1 + 80x_2 + 10x_3$,

Subject to

$100x_1 + 7x_2 + x_3 \geq 30$

$120x_1 + 10x_2 + x_3 \geq 40$

$70x_1 + 8x_2 + x_3 \geq 20$

and $x_1, x_2, x_3 \geq 0$

7. A pharmaceutical company produces two popular drugs A and B which are sold at the rate of Rs. 9.60 and Rs. 7.80, respectively. The main ingredients are x, y and z and they are

required in the following proportions :

Drugs	x%	y%	z%
A	50	30	20
B	30	30	40

The total available quantities (gm) of different ingredients are 1,600 in x, 1,400 in y and 1,200 in z. The costs (Rs) of x, y and z per gm are Rs. 8, Rs. 6 and Rs. 4, respectively. Estimate the most profitable quantities of A and B to produce, using simplex method. 8. A factory produces three different products viz. A, B and C, the profit (Rs) per unit of which are 3, 4 and 6, respectively. The products are processed in three operations viz. X, Y and Z and the time (hour) required in each operation for each unit is given below :

Operations	Products		
	A	B	C
X	4	1	6
Y	5	3	1
Z	1	2	3

The factory works 25 days in a month, at rate of 16 hours a day in two shifts. The effective working of all the processes is only 80% due to assignable causes like power cut and breakdown of machines. The factory has 3 machines in operation X, 2 machines in operation Y and one machine in operation Z. Find out the optimum product mix for the month.

9. A factory engaged in the manufacturing of pistons, rings and valves for which the profits per unit are Rs. 10, 6 and 4, respectively, wants to decide the most profitable mix. It takes one hour of preparatory work, ten hours of machining and two hours of packing and allied formalities for a piston. Corresponding requirements for rings and valves are 1, 4 and 2 , and 1, 5 and 6 hours, respectively. The total number of hours available for preparatory work, packing and allied

formalities are 100, 600 and 300, respectively. Determine the most profitable mix, assuming that what all produced can be sold.

10. A pharmaceutical company has 100 kg of material A, 180 kg of material B and 120 kg of material C available per month. They can use these materials to make three basic pharmaceutical products namely 5-10-5, 5-5-10 and 20-5-10, where the numbers in each case represent the percentage by weight of material A, material B and material C respectively, in each of the products and the balance represents inert ingredients. The cost of raw material is given below :

Ingredient	Cost per kg (Rs)
Material A	80
Material B	20
Material C	50
Inert ingredient	20

Selling price of these products is Rs. 40.50, Rs. 43 and Rs. 45 per kg respectively. There is a capacity restriction of the company for the product 5-10-5, that is, they cannot produce more than 30 kg per month. Formulate a linear programming model for maximising the monthly profit. Determine how much of each of the products should they produce in order to maximize their monthly profits.

11. The Clear-Vision Television Company manufactures models A, B and C which have profits Rs. 200, 300 and 500 per piece, respectively. According to the production license the maximum weekly production requirements are 20 for model A, 15 for B and 8 for C. The time required for manufacturing these sets is divided among following activities.

Activity	Time per piece (hours)			Total time available
	Model A	Model B	Model C	
Manufacturing	3	4	5	150
Assembling	4	5	5	200
Packaging	1	1	2	50

Formulate the production schedule as an LPP and calculate number of each model to be manufactured for yielding maximum profit.

12. A company produces two products, A and B. The sales volume of product A is at least 60 percent of the total sales of the two products. Both the products use the same raw material of which the daily availability is limited to 100 tonnes. Products A and B use this material at the rate of 2 tonnes per unit and 4 tonnes per unit, respectively. The sales price for the two products are Rs. 20 and Rs. 40 per unit.

    (a) Construct a linear programming formulation of the problem

    (b) Find the optimum solution by simplex method.

(c) Find an alternative optimum, if any.

Write the dual of the following linear programming problems.

13. Maximise $Z = 10y_1 + 8y_2 - 6y_3$

Subject to

$3y_1 + y_2 - 2y_3 \le 10$

$-2y_1 + 3y_2 - y_3 \ge 12$

$y_1, y_2, y_3 \ge 0$

14. Maximise $Z = x_1 - x_2 + x_3$

Subject to

$x_1 + x_2 + x_3 \le 10$

$2x_1 - x_3 \le 2$

$2x_1 - 2x_2 + 3x_3 \le 6$

$x_1, x_2, x_3 \ge 0$

15. Maximise $Z = 3x_1 - 2x_2$

Subject to

$x_1 \le 4$

$x_2 \le 6$

$x_1 + x_2 \le 5$

$-x_2 \le -1$

$x_1, x_2 \ge 0$

16. Minimise $Z = 4x_1 + x_2$

Subject to

$3x_1 + x_2 = 2$

$4x_1 + 3x_2 \ge 6$

$x_1 + 2x_2 \le 3$

$x_1, x_2 \ge 0$

17. Maximise $Z = 3x_1 + 4x_2 + 7x_2$

Subject to

$x_1 + x_2 + x_3 \le 10$

$4x_1 - x_2 - x_3 \ge 15$

$x_1 + x_2 + x_3 = 7$

$x_1, x_2 \ge 0$, x_3 unrestricted in sign.

18. Solve the following transportation problems :

Transportation cost (Rs./unit)

To From	W_1	W_2	W_3	Supply
F_1	6	3	2	100
F_2	4	2	3	50
Demand	60	50	40	150

19. A brick manufacturer has two depots A and B with stocks of 30,000 and 20,000 bricks respectively. He receives orders from three builders P, q and R for 11000, 20000 and 15000 bricks respectively. The distance in kms. From these depots to the builder's location are given in the following matrix :

Transportation cost (Rs./unit)

To From	A	B
P	40	20
Q	20	60
R	30	40

How should the brick manufacturer fulfill the orders so that the total transportation costs are minimised ?

Answers.

1. $x_1 = 10, x_2 = 0, Z = 70$
2. $x_1 = 0, x_2 = 5.6, x_3 = 23.2, Z = 284$
3. $x_1 = 0, x_2 = 58, Z = 760$
4. $x_1 = 8, x_2 = 0, Z = 48$
5. $x_1 = 1/5, x_2 = 6/5, x_3 = 0, Z = 210$
6. $x_1 = 1/3, x_2 = 0, x_3 =0, Z =100/3$
7. A = 2000, B = 2000, Z = 10000 8 - A = 800/7, B = 0, C = 480/7, Z = 5280/7
8. Pistons = 100/3, Rings = 200/3, valves = nil, Z = 2200/3.
10. 30, 1185, 0, Z = Rs. 20625
11. A = 50/3, B = 15, C = 8, Z = 35500/3
12. (a) max. $Z = 20x_1 + 40x_2$ subject to $2x_1 + 4x_2 \leq 100$, $-8x_1 + 24x_2 \leq 0$, $x_1,x_2 \geq 0$

    (b) $x_1 = 30, x_2 = 10, Z = 1000$

    (c) $x_1 = 10, x_2 = 0, Z = 1000$
13. Min. $G = 10x_1 + 12x_2$ subject to $3x_1 + 2x_2 \geq 10$, $x_1 - 3x_2 \geq 8$, $2x_1 - x_2 \leq 6$, $x_1,x_2 \geq 0$
14. Min $G = 10y_1 + 2y_2 + 6y_3$ Subject to $y_1 + 2y_2 + 2y_3 \geq 1$, $y_1 - 2y_3 \geq -1$, $y_1 - y_2 + 3y_3 \geq 3$, $y_1,y_2,y_3 \geq 0$

15. Min. $G = 4y_1 + 6y_2 + 5y_3 - y_4$ Subject to $y_1 + y_3 \geq 3$, $y_2 + y_3 - y_4 \geq -2$, $y_1, y_2, y_3 \geq 0$
16. Max. $G = -2y_1 + 2y_2 + 6y_5$ Subject to $4y_3 - y_4 - 3y_5 \leq 4$, $3y_3 - 2y_4 - y_5 \leq 1$, $y_3, y_4, y_5 \geq 0$
17. Min. $G = 10y_1 - 15y_2 + 7y_3$ Subject to $y_1 - 4y_2 + y_3 \geq 3$, $y_1 + y_2 + y_3 \geq 4$, $3y_1 + y_2 + y_3 = 7$, $y_1, y_2 \geq 0$, y_3 unrestricted in sign.
18. From $F_1 \rightarrow$ 10 units to w_1, 50 units to w_2 and 40 units to w_3, From $F_2 \rightarrow$ 50 units to w_1 zero units to w_2 and w_3. Total transportation cost = Rs. 490.
19. From brick depot A - Zero to P, 20000 to Q and 10000 to R. From brick depot B - 15000 to P, zero to Q and 5000 to R. Total transportation cost = Rs. 1200.

2.6. Summary. In this chapter, we find the optimum solution systematically. As we have seen in graphical method, the vertices of the feasible region gives us feasible solutions. Simplex method helps us in finding the best solution from these feasible solutions.

Books Suggested.

1. Allen, B.G.D, Basic Mathematics, Mcmillan, New Delhi.
2. Volra, N. D., Quantitative Techniques in Management, Tata McGraw Hill, New Delhi.
3. Kapoor, V.K., Business Mathematics, Sultan Chand and sons, Delhi.